Single Output Dependent Observability Normal Form

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Abstract. This paper gives the sufficient and necessary conditions which guarantee the existence of a diffeomorphism in order to transform a nonlinear system without inputs into a canonical normal form depending on its output. Moreover we extend our results to a class of systems with inputs.

1. Introduction. Since Luenberger's work [9], the design of an observer for observable linear systems with linear outputs has been a well-known concept. In order to use the same observer for nonlinear systems, the so-called observability linearization problem for nonlinear systems was born. The sufficient and necessary conditions which guarantee the existence of a diffeomorphism and of an output injection to transform a single output nonlinear system without inputs into a linear one with an output injection were firstly addressed in [12]. Then, for a multi-output nonlinear system without inputs, the linearization problem was partially solved in [13]. The complete solution to the linearization problem was given in [16]. Another approach was introduced for the analytical systems in [11] by assuming that the spectrum of the linear part must lie in the Poincaré domain and it was generalized in [14] by assuming that the spectrum of the linear part must lie in the Siegel domain. These assumptions are not in generally fulfilled. Other approaches using quadratic normal forms were given in [1] and [3]. All these approaches enable us to design an observer for a larger class of nonlinear systems.

Meanwhile, other researchers worked on designing nonlinear observers directly, such as high-gain observers [6], [4], [7]. Nevertheless, even if the conditions which guarantee the linearization method to design an observer were not generically fulfilled, this method still remained important for the nonlinear observer design first because it works well for non-analytic systems, and second because it could be used in the adaptive theory and also be useful for the observation of systems with unknown inputs. All these reasons explain why researchers carry on investigating this matter.

In [10], the author gave the sufficient and necessary geometrical conditions to transform a nonlinear system into a so-called output-dependent time scaling linear canonical form. While [5] gave the dual geometrical conditions of [10].

In this paper, as an extension of [17], we will propose a method to deduce the geometrical conditions which are sufficient and necessary to guarantee the existence of a local diffeomorphism $z = \phi(x)$ which transforms the locally observable dynamical system

$$\begin{cases} \dot{x} = f(x), \\ y = h(x), \end{cases} \tag{1.1}$$

where $x \in U \subset \mathbb{R}^n$, $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ and $h : U \subset \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth, into the following form

$$\begin{cases} \dot{z} = A(y)z + \beta(y), \\ y = z_n = Cz, \end{cases} \tag{1.2}$$

where

$$A(y) = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ \alpha_1(y) & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & \alpha_{n-2}(y) & 0 & 0 \\ 0 & \cdots & 0 & \alpha_{n-1}(y) & 0 \end{pmatrix}, \ \beta(y) = \begin{pmatrix} \beta_1(y) \\ \beta_2(y) \\ \vdots \\ \beta_{n-1}(y) \\ \beta_n(y) \end{pmatrix},$$

and $\alpha_i(y) \neq 0$ for $y \in]-a, a[$ and $a > 0$. This kind of linearization is called Single Output Dependent Observability normal form (SODO normal form).

For dynamical systems in the form (1.2), we may for example apply the following high gain observer [2]:

$$\left\{ \begin{array}{l} \dot{\hat{z}} = A(y)\hat{z} + \beta(y) - \Gamma^{-1}(y) R_\rho^{-1} C^T (C\hat{z} - y), \\ 0 = -\rho R_\rho - \bar{A}^T R_\rho - R_\rho \bar{A} + C^T C. \end{array} \right. \tag{1.3}$$

where $\Gamma(y)$ is the $n \times n$ diagonal matrix $\Gamma(y) = diag \left[\prod\limits_{i=1}^{n-1} \alpha_i(y), \prod\limits_{i=2}^{n-1} \alpha_i(y), \cdots, \alpha_{n-1}(y), 1 \right]$ and $\bar{A}$ is the $n \times n$ matrix defined as follows

$$\bar{A} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Indeed, here the output of system (1.2) is considered as an input of (1.3). Setting $e = z - \hat{z}$, the observation error can be obtained as follows:

$$\dot{e} = \left(A(y) - \Gamma^{-1}(y) R_\rho^{-1} C^T C \right) e,$$

And the convergence of such observer is proved in [2], thus in section 4 we simply highlight the design of such observer for systems in the form (1.2).

Moreover, we generalize our result to a class of systems with inputs. Then, some useful corollaries are discussed in order to deal with affine systems and the so-called left invertibility problem.

This paper is organized as follows. The next section addresses notations and technical results which are key to prove our main result. In section 3, we present our method to deduce the geometrical conditions for a nonlinear system without inputs in order to transform it into a SODO normal form. Section 4 is devoted to the generalization of our results to a class of systems with inputs. In the same section, some practical particular cases are studied, including the state affine systems and the left invertibility problem. Throughout this paper, examples are discussed in order to highlight our theoretical results.

2. Notations and technical results. Throughout this article, $L_f^{i-1}h$ for $1 \leq i \leq n$ denotes the $(i-1)^{th}$ Lie derivative of output h in the direction of f, and set $\theta_i = dL_f^{i-1}h$ as its differential. Assume that system (1.1) is locally observable, thus $\theta = \begin{pmatrix} \theta_1, & \cdots, & \theta_n \end{pmatrix}^T$ is a basis of the cotangent bundle T^*U of U. Then, we also consider the vector field τ_1 defined in [12] as follows

$$\left\{ \begin{array}{l} \theta_i(\tau_1) = 0, \ for \ 1 \leq i \leq n-1, \\ \theta_n(\tau_1) = 1, \end{array} \right. \tag{2.1}$$

and by induction we define

$$\tau_k = (-1)^{k-1} ad_f^{k-1}(\tau_1), \ for \ 2 \leq k \leq n. \tag{2.2}$$

It is clear that $\{\tau_1, \cdots, \tau_n\}$ is a basis of the tangent bundle TU of U.

Let us recall a famous result from [12].

THEOREM 2.1. *The following conditions are equivalent*

i) There exist a diffeomorphism and an output injection which transform system (1.1) into normal form (1.2) with $\alpha_k(y) = 1$ *for* $1 \le k \le n-1$.

ii) $[\tau_i, \tau_j] = 0$ *for* $1 \le i, j \le n$.

If for some $1 \le k \le n-1$ the functions $\alpha_k(y)$ in the form (1.2) are not constant, then *ii)* of Theorem 2.1 is not fulfilled. Consequently, the rest of this section is devoted to use $[\tau_i, \tau_n]$ in order to determine all the functions $\alpha_i(y)$ for $1 \le i \le n-1$.

LEMMA 2.2. *For a system in the form (1.2) we have for* $1 \le k \le n-1$,

$$\begin{aligned}\tau_k &= \tfrac{1}{\pi_k}\tfrac{\partial}{\partial z_k} + \left(A_{k-1}^k(z_n)\,z_{n-1} + \eta_{k-1}^k(z_n)\right)\tfrac{\partial}{\partial z_{k-1}} \\ &\quad + \sum_{i=1}^{k-2}\left(A_i^k(z_n)\,z_{n-k+i} + \sum_{j=n-k+i+1}^{n-1}\sum_{l=j}^{n-1} T_{j,l}^k(z_n)\,z_j z_l\right)\tfrac{\partial}{\partial z_i} \\ &\quad + \sum_{i=1}^{k-2}\left(\sum_{j=n-k+i+1}^{n}\eta_i^k(z_n)\,z_j + O_{z_n}^{[3]}(z_{n-k+i+1},\cdots,z_{n-1})\right)\tfrac{\partial}{\partial z_i},\end{aligned} \tag{2.3}$$

where $\pi_n = 1$ *and* $\pi_{k-1} = \pi_k\alpha_{k-1}$ *for* $2 \le k \le n$, $\eta_i^k(z_n)$ *and* $T_{j,l}^k(z_n)$ *are some smooth functions of* z_n, $O_{z_n}^{[3]}(z_{n-k+2},\cdots,z_{n-1})$ *represents the residue higher than order 2 with coefficient which is function of* z_n *and*

$$A_i^k(z_n) = (-1)^{k-i+1}\left(S_{k-i,1}^k\frac{\pi_i'}{\pi_i^2} + \sum_{m=k-i+1}^{k-1} S_{k-i,m-k+i+1}^k\frac{\pi_{k-m}'}{\pi_{k-m}^2}\left(\prod_{j=k-i+1}^{m}\alpha_{k-j}\right)\right)\pi_{n-k+i}, \tag{2.4}$$

where $S_{k-i,1}^k$ *and* $S_{k-i,m-k+i+1}^k$ *are defined as follows*

$$S_{j,1}^k = 1, S_{j,l}^k = S_{j-1,l}^{k-1} + S_{j,l-1}^{k-1},\ \textit{for}\ 2 \le k \le n,\ 1 \le j \le k-1\ \textit{and}\ 1 \le l \le k-j. \tag{2.5}$$

and $S_{0,l}^k = S_{i,0}^k = 0$.

Proof. For a system in the (1.2) form, equation (2.1) gives $\tau_1 = \frac{1}{\pi_1}\frac{\partial}{\partial z_1}$, then, we use equation (2.2) to obtain $\tau_2 = \frac{1}{\pi_2}\frac{\partial}{\partial z_2} + \left(\frac{\pi_1'}{\pi_1^2}\pi_{n-1}z_{n-1} + \frac{\pi_1'}{\pi_1^2}\beta_n\right)\frac{\partial}{\partial z_1}$, and

$$\begin{aligned}\tau_3 = {} & \frac{1}{\pi_3}\frac{\partial}{\partial z_3} + \left(\left(\frac{\pi_1'}{\pi_1^2}\alpha_1 + \frac{\pi_2'}{\pi_2^2}\right)\pi_{n-1}z_{n-1} + \left(\frac{\pi_1'}{\pi_1^2}\alpha_1 + \frac{\pi_2'}{\pi_2^2}\right)\beta_n\right)\frac{\partial}{\partial z_2} \\ & - \left(\frac{\pi_1'}{\pi_1^2}\pi_{n-2}z_{n-2} + \left(\frac{\pi_1'}{\pi_1^2}\pi_{n-1}\right)'\pi_{n-1}z_{n-1}^2\right)\frac{\partial}{\partial z_1} \\ & - \left(\left(\left(\frac{\pi_1'}{\pi_1^2}\pi_{n-1}\right)'\beta_n + \pi_{n-1}\beta_n'\right)z_{n-1} + \frac{\pi_1'}{\pi_1^2}\pi_{n-1}\beta_{n-1} + \beta_n\beta_n'\right)\frac{\partial}{\partial z_1}.\end{aligned}$$

Then by an induction, for $3 < k \le n$, we get

$$\begin{aligned}\tau_k = {} & \frac{1}{\pi_k}\frac{\partial}{\partial z_k} + \left(A_{k-1}^k(z_n)\,z_{n-1} + \eta_{k-1}^k(z_n)\right)\frac{\partial}{\partial z_{k-1}} \\ & + \sum_{i=1}^{k-2}\left(A_i^k(z_n)\,z_{n-k+i} + \sum_{j=n-k+i+1}^{n-1}\sum_{l=j}^{n-1} T_{j,l}^k(z_n)\,z_j z_l\right)\frac{\partial}{\partial z_i} \\ & + \sum_{i=1}^{k-2}\left(\sum_{j=n-k+i+1}^{n}\eta_i^k(z_n)\,z_j + O_{z_n}^{[3]}(z_{n-k+i+1},\cdots,z_{n-1})\right)\frac{\partial}{\partial z_i},\end{aligned}$$

where

$$A_i^k\left(z_n\right) = (-1)^{k-i+1}\left(S_{k-i,1}^k \frac{\pi_i'}{\pi_i^2} + \sum_{m=k-i+1}^{k-1} S_{k-i,m-k+i+1}^k \frac{\pi_{k-m}'}{\pi_{k-m}^2}\left(\prod_{j=k-i+1}^{m} \alpha_{k-j}\right)\right)\pi_{n-k+i},$$

with the coefficients S_i^k given by the rule (2.5).

■

In order to determine the $\alpha_i(y)$ for $1 \leq i \leq n-1$, we impose that

$$\frac{\partial}{\partial z_i} h \circ \phi^{-1} = \begin{cases} 0, & \text{for } 1 \leq i \leq n-1, \\ 1, & \text{when } i = n. \end{cases}$$

Now, we are ready to state a set of differential equations which enables us to compute functions α_i for $1 \leq i \leq n-1$.

PROPOSITION 2.3. *If there exists a diffeomorphism which transforms system (1.1) into form (1.2) then*

$$[\tau_k, \tau_n] = \lambda_k(y)\tau_k + G_n^{[1]} + R, \ for\ 1 \leq i \leq n-1,$$

where

$$G_n^{[1]} = \sum_{i=1}^{k-2}\left(\frac{1}{\pi_k} T_{k,n-k+i}^k z_{n-k+i}\right)\frac{\partial}{\partial z_i} + \frac{1}{\pi_k} T_{k,k}^k z_k \frac{\partial}{\partial z_{2k-n}},$$

and

$$R = \sum_{i=1}^{k-2}\left(\sum_{j=n-k+i+1}^{n} \bar{\eta}_i^k\left(z_n\right) + O_{z_n}^{[2]}\left(z_{n-k+i+1}, \cdots, z_{n-1}\right)\right)\frac{\partial}{\partial z_i}$$

and

$$\lambda_k(y) = diag\{\delta_1^k(y), \cdots, \delta_i^k(y), \cdots, \delta_k^k(y), 0, \cdots, 0\}, \ for\ 1 \leq i \leq k-1, \tag{2.6}$$

where $\delta_k^k = A_k^n + \frac{\pi_k'}{\pi_k}$ *and* $\delta_i^k = A_i^n - A_{n-k+i}^n - \frac{\left(A_i^k\right)'}{A_i^k}$ *for* $1 \leq i \leq k-1$, *and* A_i^k *is given in (2.4).*

Proof. According to equation (2.3), for $1 \leq k \leq n-1$ we have

$$\begin{aligned}
[\tau_k, \tau_n] = & \left(A_k^n + \frac{\pi_k'}{\pi_k}\right)\frac{1}{\pi_k}\frac{\partial}{\partial z_k} \\
& + \sum_{i=1}^{k-2}\left(\left(A_i^n - A_{n-k+i}^n - \frac{\left(A_i^k\right)'}{A_i^k}\right) A_i^k z_{n-k+i} + \frac{1}{\pi_k} T_{k,n-k+i}^k z_{n-k+i}\right)\frac{\partial}{\partial z_i} + \frac{1}{\pi_k} T_{k,k}^k z_k \frac{\partial}{\partial z_{2k-n}} \\
& + \sum_{i=1}^{k-2}\left(\sum_{j=n-k+i+1}^{n} \bar{\eta}_i^k\left(z_n\right) + O_{z_n}^{[2]}\left(z_{n-k+i+1}, \cdots, z_{n-1}\right)\right)\frac{\partial}{\partial z_i}.
\end{aligned}$$

Set $\lambda_k(y) = diag\{\delta_1^k(y), \cdots, \delta_i^k(y), \cdots, \delta_k^k(y), 0, \cdots, 0\}$, where $\delta_k^k = A_k^n + \frac{\pi_k'}{\pi_k}$ and $\delta_i^k = A_i^n - A_{n-k+i}^n - \frac{\left(A_i^k\right)'}{A_i^k}$ for $1 \leq i \leq k-1$, then

$$[\tau_k, \tau_n] = \lambda_k(y)\tau_k + G_n^{[1]} + R. \tag{2.7}$$

■

REMARK 1. *In equation (2.7), $\lambda_k(y)$ could be uniquely determined since $G_n^{[1]}$ might be separated according to the coefficients of second-order terms in τ_n.*

Finally, the following result enables us to determine all the functions $\alpha_i(y)$ for all $1 \leq i \leq n-1$.

PROPOSITION 2.4. *If there exists a diffeomorphism which transforms system (1.1) into form (1.2), then* $\alpha_i = \frac{\pi_i}{\pi_{i+1}}$ *for* $1 \leq i \leq n-2$, *and* $\alpha_{n-1} = \pi_{n-1}$, *where*

$$\begin{cases} \pi_i = c_i \exp\left[\int \left(\exp \int \left(\delta_i^i - \delta_i^{n-1} - \delta_{i+1}^{i+1}\right) dy - \bar{B}_i^{n-1}\right) dy\right], \ for\ 1 \leq i \leq n-2, \\ \pi_{n-1} = c_{n-1} \exp\left(\int \left(\frac{\delta_{n-1}^{n-1} - \bar{A}_{n-1}^n}{2}\right) dy\right), \end{cases} \tag{2.8}$$

with $\bar{B}_1^k = 0$ *and for* $1 \leq i, k \leq n-1$ *and* $1 \leq i \leq n-1$

$$\bar{B}_i^k = \sum_{m=k-i+1}^{k-1} S_{k-i,m-k+i+1}^k \frac{\pi'_{k-m}}{\pi_{k-m}}. \tag{2.9}$$

Proof. Define

$$B_i^k = \frac{\pi'_i}{\pi_i} + \bar{B}_i^k. \tag{2.10}$$

According to equation (2.4), for $1 \leq i, k \leq n-1$,

$$\frac{\left(A_i^k\right)'}{A_i^k} = \frac{\left(B_i^k\right)'}{B_i^k} - \frac{\pi'_i}{\pi_i} + \frac{\pi'_{n-k+i}}{\pi_{n-k+i}}.$$

As $\delta_k^k = A_k^n + \frac{\pi'_k}{\pi_k}$, hence

$$\delta_i^{n-1} = A_i^n - A_{1+i}^n - \frac{\left(B_i^{n-1}\right)'}{B_i^{n-1}} + \frac{\pi'_i}{\pi_i} - \frac{\pi'_{1+i}}{\pi_{1+i}} = \delta_i^i - \delta_{1+i}^{1+i} - \left(\frac{\pi'_i}{\pi_i} + \bar{B}_i^{n-1}\right)' / \left(\frac{\pi'_i}{\pi_i} + \bar{B}_i^{n-1}\right).$$

which yields

$$\pi_i = c_i \exp\left[\int \left(\exp \int \left(\delta_i^i - \delta_i^{n-1} - \delta_{1+i}^{1+i}\right) dy - \bar{B}_i^{n-1}\right) dy\right], \ for\ 1 \leq i \leq n-2,$$

where $\bar{B}_i^{n-1}$ is defined in (2.9) and $c_i \in R$, $c_i \neq 0$.

As $\delta_{n-1}^{n-1} = 2\frac{\pi'_{n-1}}{\pi_{n-1}} + \bar{A}_{n-1}^n$, where $\bar{A}_{n-1}^n = \sum_{m=2}^{n-1} S_{1,m}^n \frac{\pi'_{n-m}}{\pi_{n-m}}$, then

$$\pi_{n-1} = c_{n-1} \exp\left(\int \left(\frac{\delta_{n-1}^{n-1} - \bar{A}_{n-1}^n}{2}\right) dy\right).$$

■

REMARK 2. *For system (1.2), if we set* $\alpha_i = s(y)$ *for* $1 \leq i \leq n-1$, *then*

$$\delta_{n-1}^{n-1} = A_{n-1}^n + \frac{\pi'_{n-1}}{\pi_{n-1}} = 2\frac{\pi'_{n-1}}{\pi_{n-1}} + \sum_{i=2}^{n-1} \frac{\pi'_{n-i}}{\pi_{n-i}}.$$

By the definition of π_i *for* $1 \leq i \leq n-1$, *we have* $\pi_k = s^{n-k}$ *for* $1 \leq k \leq n-1$, *therefore*

$$\delta_{n-1}^{n-1} = 2\frac{s'}{s} + \sum_{i=2}^{n-1} i\frac{s'}{s} = l_n \frac{s'}{s},$$

where $l_n = \frac{n(n-1)}{2} + 1$. *In such a way, we obtain the same result as the one stated in [10].*

3. Main result. If there exists a diffeomorphism which transforms system (1.1) into form (1.2), then equation (2.8) of Proposition 2.4 gives all α_i for $1 \le i \le n-1$. Therefore, let us consider a new family of vector fields defined as follows:

$$\widetilde{\tau}_1 = \pi_1 \tau_1 \textit{ and } \widetilde{\tau}_{i+1} = \frac{1}{\alpha_i}[\widetilde{\tau}_i, f], \textit{ for } 1 \le i \le n-1. \tag{3.1}$$

Set

$$\theta(\widetilde{\tau}_1, \cdots, \widetilde{\tau}_n) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \vdots & \cdots & \pi_{n-1} & \widetilde{l}_{2,n} \\ \vdots & \ldots & \ddots & \ldots & \vdots \\ \vdots & \pi_2 & \cdots & \cdots & \vdots \\ \pi_1 & \widetilde{l}_{n,2} & \cdots & \cdots & \widetilde{l}_{n,n} \end{pmatrix} := \widetilde{\Lambda},$$

where

$$\widetilde{l}_{k,j} = \theta_k(\widetilde{\tau}_j) \textit{ for } 2 \le k \le n \textit{ and } n-k+2 \le j \le n.$$

Consider the following R^n-valued form ω

$$\omega = \widetilde{\Lambda}^{-1}\theta := (\omega_1, \omega_2, \cdots, \omega_n)^T, \tag{3.2}$$

where, for $1 \le s \le n$, we have

$$\omega_s = \sum_{m=1}^{n} r_{s,m}\theta_m. \tag{3.3}$$

Then, the following algorithm gives all the components of ω.

ALGORITHM 1.

$$\begin{aligned} &\textit{for } 1 \le j \le n, \\ &\qquad r_{n,j} = \cdots = r_{n-j+2,j} = 0 \textit{ and } r_{n-j+1,j} = 1. \\ &\textit{for } 2 \le k \le n-1 \textit{ and } 1 \le j \le n, \\ &\qquad r_{n-k,j} = -\sum_{i=2}^{k} \widetilde{l}_{k,n-k+i-(j-1)} r_{n-k+i-(j-1),j}, \end{aligned}$$

and then, equation (3.3) becomes: $\omega_s = \sum_{m=1}^{n-s+1} r_{s,m}\theta_m$.

THEOREM 3.1. *The following conditions are equivalent*

1) *There exists a diffeomorphism which transforms system (1.1) into a SODO normal form (1.2).*

2) *There exists a family of functions $\alpha_i(y)$ for $1 \le i \le n-1$ such that the family of vector fields $\widetilde{\tau}_i$ for $1 \le i \le n$ defined in (3.1) satisfies the following commutativity conditions*

$$[\widetilde{\tau}_i, \widetilde{\tau}_j] = 0, \textit{ for } 1 \le i, j \le n. \tag{3.4}$$

3) *There exists a family of functions $\alpha_i(y)$ for $1 \le i \le n-1$ such that the R^n-valued form ω defined in (3.2) satisfies the following condition*

$$d\omega = 0. \tag{3.5}$$

Proof. Assume that there exists a diffeomorphism which transforms system (1.1) into form (1.2), then we compute $\alpha_i(y)$ for $1 \leq i \leq n-1$ from equation (2.8) in Proposition 2.4. Thus, it is easy to show that $\tau_1 = \frac{1}{\pi_1}\frac{\partial}{\partial z_1}$ which yields that $\widetilde{\tau}_1 = \frac{\partial}{\partial z_1}$ and then, by construction we obtain $\widetilde{\tau}_i = \frac{\partial}{\partial z_i}$ for $2 \leq i \leq n$. Consequently, we have $[\widetilde{\tau}_i, \widetilde{\tau}_j] = 0$ for $1 \leq i, j \leq n$.

Reciprocally, assume that there exist $\alpha_i > 0$ for $1 \leq i \leq n-1$ such that $[\widetilde{\tau}_i, \widetilde{\tau}_j] = 0$ for $1 \leq i, j \leq n$, then it is well-known ([8], [15]) that we can find a local diffeomorphism $\phi = z$ such that

$$\phi_*(\widetilde{\tau}_i) = \frac{\partial}{\partial z_i}.$$

As $\phi_*(\widetilde{\tau}_i) = \frac{\partial}{\partial z_i}$ is constant, hence

$$\frac{\partial}{\partial z_i}\phi_*(f) = \phi_*\left([\widetilde{\tau}_i, f]\right) = \alpha_i \phi_*(\widetilde{\tau}_{i+1}) = \alpha_i \frac{\partial}{\partial z_{i+1}},$$

thus $\frac{\partial}{\partial z_i}\phi_*(f) = \alpha_i \frac{\partial}{\partial z_{i+1}}$ for $1 \leq i \leq n-1$. Consequently, by integration we obtain: $\phi_*(f) = A(y)z + \beta(y)$.

Moreover, as $dh \circ \widetilde{\tau}_i = 0$ for $1 \leq i \leq n-1$ and $dh \circ \widetilde{\tau}_n = 1$, we obtain $h \circ \phi^{-1} = z_n$.

Finally, in order to prove that in Theorem (3.1) Condition 2) is equivalent to Condition 3), it is sufficient to prove that equation (3.4) is equivalent to equation (3.5).

Recall that for any two vector fields X, Y, we have

$$d\omega(X,Y) = L_X\left(\omega(Y)\right) - L_Y\left(\omega(X)\right) - \omega([X,Y]).$$

Setting $X = \widetilde{\tau}_i$ and $Y = \widetilde{\tau}_j$, we obtain

$$d\omega(\widetilde{\tau}_i, \widetilde{\tau}_j) = L_{\widetilde{\tau}_i}\omega(\widetilde{\tau}_j) - L_{\widetilde{\tau}_j}\omega(\widetilde{\tau}_i) - \omega([\widetilde{\tau}_i, \widetilde{\tau}_j]).$$

As $\omega(\widetilde{\tau}_j)$ and $\omega(\widetilde{\tau}_i)$ are constant, then we have

$$d\omega(\widetilde{\tau}_i, \widetilde{\tau}_j) = -\omega([\widetilde{\tau}_i, \widetilde{\tau}_j]).$$

Because ω is an isomorphism and $(\widetilde{\tau}_i)_{1\leq i\leq n}$ is a basis of TU, then equation (3.4) is equivalent to equation (3.5). ∎

REMARKS 1. *i) The R^n-valued form ω can be viewed as an isomorphism $TU^n \to U \times I\!R^n$ which brings each $\widetilde{\tau}_i$ to the canonical vector basis $\frac{\partial}{\partial z_I}$. Moreover, $d\omega = 0$ means that there is a local diffeomorphism $\phi : U \to U$ such that ω is the tangent map of ϕ.*

ii) The diffeomorphism $\phi(x) = z$ is determined by $\omega = \phi_*(x)$, which can be given locally as follows

$$z_i = \phi_i(x) = \int_\gamma \omega_i + \phi_i(0) \; for \; 1 \leq i \leq n,$$

where γ is a smooth path from 0 to x lying in a neighborhood $V_0 \subseteq U$ of 0.

The following simple example is studied in order to illustrate Theorem 3.1.

EXAMPLE 1. *Let us consider the following system*

$$\begin{cases} \dot{x}_1 = \frac{\gamma(y)}{1+x_4}x_1x_3, \\ \dot{x}_2 = \frac{\beta(y)}{1+x_4}x_1, \\ \dot{x}_3 = \mu(y)x_2, \\ \dot{x}_4 = \gamma(y)x_3, \\ y = x_4. \end{cases} \tag{3.6}$$

which gives

$$\begin{cases} \theta_1 = dx_4, \\ \theta_2 = \gamma dx_3 + \gamma' x_3 dx_4, \\ \theta_3 = \gamma\mu dx_2 + 2\gamma'\gamma x_3 dx_3 + \left((\gamma\mu)' x_2 + (\gamma'\gamma)' x_3^2\right) dx_4, \\ \theta_4 = \gamma\mu\frac{\beta}{1+x_4}dx_1 + \left(2\gamma'\mu + (\gamma\mu)'\right)\gamma x_3 dx_2 \\ \quad + \left(2\gamma'\gamma\mu x_2 + \gamma\left(\gamma\mu\right)' x_2 + 3\gamma\left(\gamma'\gamma\right)' x_3^2\right) dx_3 + O^{[2]}\left(x_1,\ x_2,\ x_3\right)\theta_1. \end{cases}$$

Then we have $\tau_1 = \frac{1+x_4}{\gamma\mu\beta}\frac{\partial}{\partial x_1}$. *Consequently we obtain*

$$\left\{\begin{array}{l} \tau_2 = \frac{1}{\gamma\mu}\frac{\partial}{\partial x_2} + (1+x_4)\,\gamma\frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)^2}x_3\frac{\partial}{\partial x_1}, \\ \tau_3 = \frac{1}{\gamma}\frac{\partial}{\partial x_3} - \gamma\mu\,(1+x_4)\,\frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)^2}x_3\frac{\partial}{\partial x_2} + \left(\frac{(\gamma\mu)'}{(\gamma\mu)^2} + \beta\frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)^2}\right)\gamma x_2\frac{\partial}{\partial x_1} + R_{1,3}\tau_1, \\ \tau_4 = \frac{\partial}{\partial x_4} + \left(\frac{\gamma'}{\gamma} + \frac{(\gamma\mu)'}{(\gamma\mu)} + \frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)}\right)x_3\frac{\partial}{\partial x_3} - \left(\frac{(\gamma\mu)'}{(\gamma\mu)} + 2\frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)}\right)x_2\frac{\partial}{\partial x_2} \\ \quad + \left(\frac{1}{1+x_4} + \frac{(\gamma\mu\beta)'}{\gamma\mu\beta}\right)x_1\frac{\partial}{\partial x_1} + R_{1,4}\,(z_3, z_2)\,\tau_1 + R_{2,3}(z_3^2)\tau_2. \end{array}\right.$$

A straightforward computation gives

$$\begin{aligned} &\delta_1^1 = 2\frac{(\gamma\mu\beta)'}{\gamma\mu\beta},\ \delta_2^2 = -2\frac{(\gamma\mu\beta)'}{\gamma\mu\beta},\ \delta_3^3 = 2\frac{\gamma'}{\gamma} + \frac{(\gamma\mu)'}{\gamma\mu} + \frac{(\gamma\mu\beta)'}{\gamma\mu\beta}, \\ &\delta_1^3 = 4\frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)} - \left[\frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)}\right]' / \left[\frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)}\right], \\ &\delta_2^3 = -\left(2\frac{\gamma'}{\gamma} + \frac{(\gamma\mu)'}{\gamma\mu} + 3\frac{(\gamma\mu\beta)'}{\gamma\mu\beta}\right) - \left(\frac{(\gamma\mu)'}{\gamma\mu} + \frac{(\gamma\mu\beta)'}{\gamma\mu\beta}\right)' / \left(\frac{(\gamma\mu)'}{\gamma\mu} + \frac{(\gamma\mu\beta)'}{\gamma\mu\beta}\right). \end{aligned}$$

According to equation (2.8) in Proposition 2.4, we obtain

$$\left\{\begin{array}{l} \pi_1 = c_1\exp\left[\int\left(\exp\int\left(\delta_1^1 - \delta_1^3 - \delta_2^2\right)dy\right)dy\right] = c_1\gamma\mu\beta, \\ \pi_2 = c_2\exp\left[\int\left(\exp\int\left(\delta_2^2 - \delta_2^3 - \delta_3^3\right)dy - \frac{\pi_1'}{\pi_1}\right)dy\right] = c_2\gamma\mu, \\ \pi_3 = c_3\exp\left(\int\left(\frac{1}{2}\left(\delta_3^3 - \frac{\pi_1'}{\pi_1} - \frac{\pi_2'}{\pi_2}\right)\right)dy\right) = c_3\gamma. \end{array}\right.$$

Thus $\alpha_1 = \frac{\pi_1}{\pi_2} = \frac{c_1}{c_2}\beta$, $\alpha_2 = \frac{\pi_2}{\pi_3} = \frac{c_2}{c_3}\mu$ *and* $\alpha_3 = \frac{\pi_3}{\pi_4} = c_3\gamma$, *so the new vector fields are*

$$\widetilde{\tau}_1 = c_1\,(1+x_4)\,\frac{\partial}{\partial x_1},\ \widetilde{\tau}_2 = c_2\frac{\partial}{\partial x_2},\ \widetilde{\tau}_3 = c_3\frac{\partial}{\partial x_3},\ \widetilde{\tau}_4 = \frac{\partial}{\partial x_4} + \frac{x_1}{1+x_4}\frac{\partial}{\partial x_1}.$$

It is clear that $[\widetilde{\tau}_i, \widetilde{\tau}_j] = 0$ *for all* $1 \le i, j \le 4$. *Therefore, according to Theorem 3.1, system (3.6) can be transformed into SODO normal form (1.2).*

Moreover as

$$\widetilde{\Lambda} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \gamma & \gamma' x_3 \\ 0 & \gamma\mu & 2\gamma'\gamma x_3 & (\gamma\mu)'\,x_2 + 2\,(\gamma'\gamma)'\,x_3^2 \\ \gamma\mu\beta & \left(2\gamma'\mu + (\gamma\mu)'\right)\gamma x_3 & 2\gamma'\gamma\mu x_2 + \gamma\,(\gamma\mu)'\,x_2 + 6\gamma\,(\gamma'\gamma)'\,x_3^2 & \gamma\frac{x_1}{(1+x_4)^2}\mu\beta + R \end{pmatrix},$$

where $R = O_{x_4}^{[2]}(x_1,\ x_2,\ x_3)$, *a straightforward computation gives*

$$\omega = \widetilde{\Lambda}^{-1}\theta = \left(\begin{array}{cccc} d\frac{x_1}{c_1(1+x_4)}, & d\left(\frac{x_2}{c_2}\right), & d\left(\frac{x_3}{c_3}\right), & dx_4 \end{array}\right)^T.$$

As $\omega = d\phi$, *thus the diffeomorphism which transforms system (3.6) into SODO normal form (1.2) is*

$$\phi(x) = z = (\frac{x_1}{c_1\,(1+x_4)}, \frac{x_2}{c_2}, \frac{x_3}{c_3}, x_4)^T.$$

with which system (3.6) could be transformed into

$$\left\{\begin{array}{l} \dot{z}_1 = 0, \\ \dot{z}_2 = \frac{c_1}{c_2}\beta(y)z_1, \\ \dot{z}_3 = \frac{c_2}{c_3}\mu(y)z_2, \\ \dot{z}_4 = c_3\gamma(y)z_3. \end{array}\right.$$

So far, in this paper, we have only considered systems without inputs. The next section is devoted to systems that are also driven by an input term.

4. Extension to systems with inputs. Consider a system with inputs in the following form

$$\left\{\begin{array}{l} \dot{x} = f(x) + g(x,u), \\ y = h(x), \end{array}\right. \tag{4.1}$$

where $x \in U \subset \mathbb{R}^n$, $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, $g : U \times \mathbb{R}^m \to \mathbb{R}^n$, $h : U \subset \mathbb{R}^n \to \mathbb{R}$ are analytic functions and for $x \in U$, $g(x,0) = 0$.

For system (4.1), the SODO normal form along its output trajectory $y(t)$ is as follows

$$\left\{\begin{array}{l} \dot{z} = A(y)z + \beta(y) + \eta(y,u), \\ y = z_n = Cz, \end{array}\right. \tag{4.2}$$

where $A(y)$ and $\beta(y)$ are given in (1.2) and $\eta(y,u) = \left[\begin{array}{cccc} \eta_1(y,u), & \eta_2(y,u), & \cdots, & \eta_n(y,u) \end{array}\right]^T$.

THEOREM 4.1. *System (4.1) can be transformed into SODO normal form (4.2) by a diffeomorphism if and only if*

i) one of conditions in Theorem 3.1 is fulfilled.

ii) $[g, \widetilde{\tau}_i] = 0$ *for* $1 \leq i \leq n-1$.

Proof. From Theorem 3.1, we can state that there exists a diffeomorphism ϕ such that

$$\phi_*(f) = A(y)z + \beta(y).$$

For $1 \leq i \leq n-1$, because $\phi_*\left(\widetilde{\tau}_i\right) = \frac{\partial}{\partial z_i}$ is constant, hence we have

$$\frac{\partial}{\partial z_i}\phi_*(g) = \phi_*([g, \widetilde{\tau}_i]) = 0.$$

Therefore $\phi_*(g) = \eta(y,u)$. Thus, we obtain the form (4.2).

■

REMARK 3. *If* $g(x,u) = g_1(x)u_1 + \cdots + g_m(x)u_m$, *and also both conditions i) and ii) of Theorem 4.1 are fulfilled, then*

$$\eta(y,u) = B_1(y)u_1 + \cdots + B_m(y)u_m.$$

Let us now study some special cases of the term $\eta(y,u)$.

COROLLARY 4.2. *Assume that conditions i) and ii) of Theorem 4.1 are fulfilled,*

a) if $[g, \widetilde{\tau}_n] = 0$, *then*

$$\eta(y,u) = \eta(u).$$

b) if $g(x,u) = g_1(x)u_1 + \cdots + g_m(x)u_m$ *and*

$$[g_k, \widetilde{\tau}_i] = 0, \; for \; 1 \leq i \leq n \; and \; 1 \leq k \leq m,$$

then

$$\eta(y,u) = B_1u_1 + \cdots + B_mu_m,$$

where B_i *are constant vector fields.*

EXAMPLE 2. *Let us consider the following system*

$$\left\{\begin{array}{l} \dot{x}_1 = \frac{\gamma(y)}{1+x_3}x_1x_2 + \frac{x_1}{1+x_3}u, \\ \dot{x}_2 = \frac{\mu(y)}{1+x_3}x_1, \\ \dot{x}_3 = \gamma(y)x_2 + u, \\ y = x_3. \end{array}\right. \tag{4.3}$$

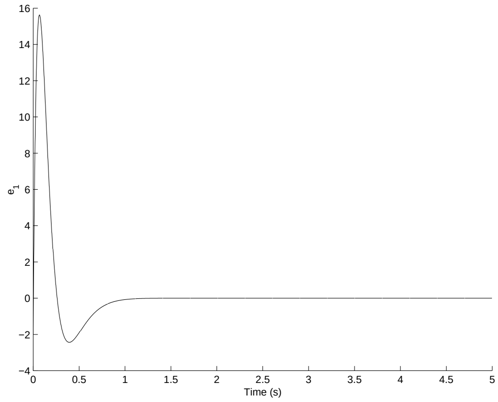

FIG. 4.1. *Observation error between z_1 and $\hat{z}_1$*

A straightforward computation gives:

$$\tau_1 = \frac{1+x_3}{\gamma\mu}\frac{\partial}{\partial x_1},\ \tau_2 = \frac{1}{\gamma}\frac{\partial}{\partial x_2} + \left((1+x_3)\frac{(\gamma\mu)'}{\gamma\mu^2}\right)x_2\frac{\partial}{\partial x_1},$$
$$\tau_3 = \frac{\partial}{\partial x_3} + \left(\frac{(\mu\gamma)'}{(\mu\gamma)} + \frac{\gamma'}{\gamma}\right)x_2\frac{\partial}{\partial x_2} + \left(\frac{1}{1+x_3} - \frac{(\gamma\mu)'}{\gamma\mu}\right)x_1\frac{\partial}{\partial x_1}.$$

Then, we obtain

$$\delta_1^1 = 0,\ \delta_2^2 = 2\frac{\gamma'}{\gamma} + \frac{(\mu\gamma)'}{(\mu\gamma)},\ \delta_1^2 = -\frac{(\mu\gamma)'}{(\mu\gamma)} - 2\frac{\gamma'}{\gamma} - \left(\frac{(\mu\gamma)'}{(\mu\gamma)}\right)' / \left(\frac{(\mu\gamma)'}{(\mu\gamma)}\right).$$

Then, according to (2.8), we have

$$\begin{cases} \pi_1 = c_1 \exp\left[\int \left(\exp \int \left(\delta_1^1 - \delta_1^2 - \delta_2^2\right) dy\right) dy\right] = c_1\gamma\mu, \\ \pi_2 = c_2 \exp\left(\int \left(\frac{1}{2}\left(\delta_2^2 - \frac{\pi_1'}{\pi_1}\right)\right) dy\right) = c_2\gamma. \end{cases}$$

which yields $\alpha_1(y) = \frac{c_1}{c_2}\mu(y)$ *and* $\alpha_2(y) = c_2\gamma(y)$. *Therefore, we obtain* $\widetilde{\tau}_1 = c_1(1+x_3)\frac{\partial}{\partial x_1}$, $\widetilde{\tau}_2 = c_2\frac{\partial}{\partial x_2}$ *and* $\widetilde{\tau}_3 = \frac{\partial}{\partial x_3} + \frac{x_1}{1+x_3}\frac{\partial}{\partial x_1}$.

As $g = \frac{x_1}{1+x_3}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} = \widetilde{\tau}_3$ *then* $[g, \widetilde{\tau}_1] = [g, \widetilde{\tau}_2] = 0$ *and system (4.3) is transformed into*

$$\begin{cases} \dot{z}_1 = 0, \\ \dot{z}_2 = \frac{c_1}{c_2}\mu(y) z_1, \\ \dot{z}_3 = c_2\gamma(y) z_2 + u, \\ y = z_3. \end{cases} \tag{4.4}$$

by the following diffeomorphism

$$\phi(x) = z = \left(\frac{x_1}{c_1(1+x_3)}, \frac{x_2}{c_2}, x_3\right)^T.$$

Following the proposed high gain observer in the form (1.3), the corresponding observer for the system (4.4) can be designed as follows:

$$\begin{cases} \dot{\hat{z}}_1 = -\frac{\rho^3}{\gamma\mu}(\hat{z}_3 - z_3), \\ \dot{\hat{z}}_2 = \frac{c_1}{c_2}\mu(y)\hat{z}_1 - 3\frac{\rho^2}{\gamma}(\hat{z}_3 - z_3), \\ \dot{\hat{z}}_3 = c_2\gamma(y)\hat{z}_2 - 3\rho(\hat{z}_3 - z_3) + u, \end{cases}$$

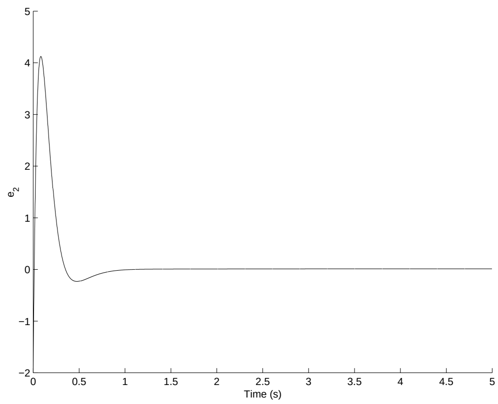

FIG. 4.2. *Observation error between z_2 and $\hat{z}_2$*

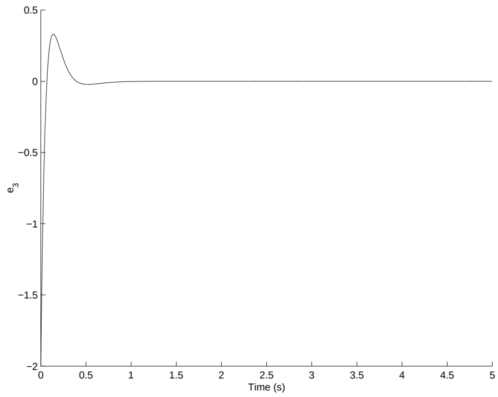

FIG. 4.3. *Observation error between z_3 and $\hat{z}_3$*

where ρ is the tunable gain. For a more specific but simple simulation, choose $c_1 = c_2 = 1$, $u(t) = 1$, $\mu(y) = 1 + y^2$, and $\gamma(y) = 2 + cos(y)$. Its simulation results are presented in Fig. 4.1, Fig. 4.2 and Fig. 4.3 which respectively present the convergence of system's states and their estimations.

In addition, in order to solve the left invertibility problem, the Observability Matching Condition (OMC) for system (4.1) with $m = 1$ is as follows

$$\begin{cases} L_g L_f^{i-1} h = 0, \forall x \in U,\ 1 \leq i \leq n-1, \\ L_g L_f^{n-1} h \neq 0. \end{cases}$$

COROLLARY 4.3. *Assume conditions i) and ii) of Theorem 4.1 are fulfilled and the OMC is verified then*

$$\eta(y,u) = \begin{bmatrix} \eta_1(y,u), & 0, & \cdots, & 0 \end{bmatrix}^T.$$

REMARK 4. *The OMC for system (4.1) with $m = 1$ is equivalent to $g \in span\{\widetilde{\tau}_1\}$.*

We give another example in order to highlight Corollary 4.3.

EXAMPLE 3. *Consider the following system*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = \mu(y)x_1 + \mu(y)x_1^2 + \frac{x_2}{1+x_1}u, \\ \dot{x}_3 = \gamma(y)\frac{x_2}{1+x_1}, \\ y = x_3. \end{cases} \tag{4.5}$$

A straightforward computation gives $\tau_1 = \frac{1}{\gamma\mu}\frac{\partial}{\partial x_1} + \frac{1}{\gamma\mu}\frac{x_2}{1+x_1}\frac{\partial}{\partial x_2}$. *From equation (2.8)in Proposition 2.4, we can determine* $\alpha_1(y) = \frac{c_1}{c_2}\mu(y)$ *and* $\alpha_2(y) = c_2\gamma(y)$. *Thus, we have* $\widetilde{\tau}_1 = c_1\frac{\partial}{\partial x_1} + c_1\frac{x_2}{1+x_1}\frac{\partial}{\partial x_2}$, $\widetilde{\tau}_2 = c_2(1+x_1)\frac{\partial}{\partial x_2}$ *and* $\widetilde{\tau}_3 = \frac{\partial}{\partial x_3}$.

As $g \in span\{\widetilde{\tau}_1\}$, *then the OMC condition is fulfilled, therefore system (4.5) could be transformed by the following diffeomorphism*

$$\phi(x) = z = \left(\frac{x_1}{c_1}, \frac{x_2}{c_2(1+x_1)}, x_3\right)^T.$$

into

$$\begin{cases} \dot{z}_1 = \frac{u}{c_1}, \\ \dot{z}_2 = \frac{c_1}{c_2}\mu(y) z_1, \\ \dot{z}_3 = c_2\gamma(y) z_2, \\ y = z_3. \end{cases}$$

5. Conclusion. In this paper, we have put forward the geometrical conditions which allow us to determine whether a nonlinear system can be transformed locally into the SODO normal form by means of a diffeomorphism and of an output injection. In our main result we state two equivalent ways to check these conditions. In the first one, we have used Lie brackets commutativity and the second one was based on the one-forms. Moreover, an extension of our results is stated for a class of nonlinear systems with inputs.

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